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AN APLANATIC ANASTIGMATIC LENS SYSTEM SUITABLE FOR ASTROGRAPH OBJECTIVES

BY

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1. As is well known the ordinary Fraunhofer lens is characterized by a considerable astigmatism and curvature of field. In consequence the usable angular field is rather limited with this type of objective lens.

In the Petzval lens system, consisting of four separate lenses, the astigmatism and curvature of field are much smaller than for the Fraunhofer lens. For the Cooke triplet lens these aberrations are still further reduced.

K. SCHWARZSCHILD[1] in a systematic analysis of lens systems gives the following numerical data. For the Fraunhofer lens the aberration disc is an ellipse the major axis of which is $104'' OG^2$, while the minor axis is $47'' OG^2$. Here O means the aperture ratio expressed with the aperture ratio 1 : 10 as unit, while G is the angular diameter of the field, with 6° as unit. For the Petzval lens the aberration disc is approximately a circle with diameter $12'' O^2G$, while for the Cooke triplet lens it is approximately a circle with diameter $3'' O^2G$.

The objective of the standard Carte du Ciel astrograph is a Fraunhofer lens of 34 cm. aperture and 3.4 m. focal length, corresponding to $O = 1$. The standard field employed is $2° \times 2°$, i. e. $G = 0.47$. The maximum extent of the aberrational disc is thus equal to $23''$.

The following data, which refer to the Bruce telescope at Bloemfontein (formerly Arequipa), are typical of an astrograph with a Petzval objective lens, namely, aperture 60 cm., focal length 3.4 m., standard field $7° \times 6°$, i. e. $O = 1.7$, $G = 1.5$. The corresponding maximum extent of the aberrational disc calculated from K. SCHWARZSCHILD's formula given above is about $50''$. For a triplet with similar data the result would be about $12''$.

In judging these results a number of circumstances should be observed. With regard to the Fraunhofer lens the effect of the secondary spectrum must be considered. The combined effect of secondary spectrum, astigmatism, and curvature of field is a complicated distribution of light in the focal plane, or rather the plane of the photographic plate, which is close to the focal plane but not necessarily coinciding with it. This light distribution is modified, especially in its central parts, by diffraction, by the influence of atmospheric disturbance (tremor), and by the diffusion of light in the film of the photographic plate. In the case of the fainter stars recorded only the central part of the light distribution affects the photographic plate.

The result is that the images of the fainter stars appear much smaller than one would expect from the calculated effect of astigmatism and curvature of field alone.

Thus the effect of the geometrical aberrations considered is only partly visible in the form of reduced sharpness of the photographic images. A certain unhomogeneity of the field in a photometric respect is produced in addition.

With regard to the Petzval and Cooke triplet lenses the situation is similar. However, the effect of lens aberrations of the fifth and higher orders must be considered here. In a well designed lens these partly balance the third order aberrations.

The curvature of the most heavily curved lens surface of a Fraunhofer lens (apart from the inner surfaces, which need not be considered in this connection since their curvatures are nearly equal, and the refracting indices not very different) is about 1.7 in units of the reciprocal of the focal length. For the Petzval lens the corresponding maximum curvature according to K. Schwarzschild is 2.0, and for the Cooke triplet lens it is about 4.2. This shows that the Petzval type lends itself well to constructions with large aperture ratio, while for the Cooke triplet type the aperture ratio is more restricted. It is, in fact, well-known that the Petzval type is preferable for large aperture and moderate field, while the Cooke triplet type is preferable for moderate aperture and large field.

Finally, it may be mentioned that the triplet lens has the drawback of a secondary spectrum about 1.4 times that of the Fraunhofer lens, while for the Petzval lens the corresponding factor is about 0.9.

In the Fraunhofer objective lens the two component lenses are practically in contact. The Petzval and the Cooke triplet objectives on the other hand are characterized by a relatively wide separation of the component lenses. In the case of the Petzval objective the distance between front and rear lenses is about 0.4 times the focal length f. For the Cooke triplet objective the corresponding distance is about 0.25 f.

The relative positions of the component lenses of Petzval and triplet objectives must be adjusted to a high degree of accuracy, and this adjustment must be maintained to obtain a

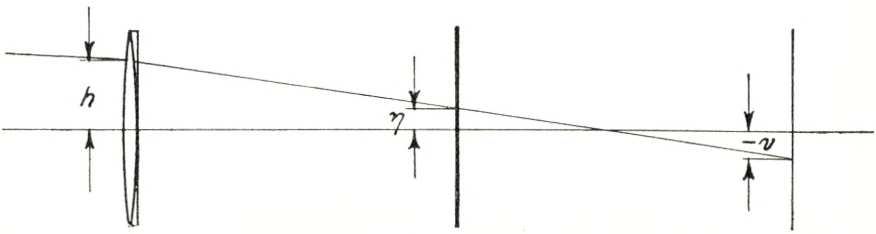

Fig. 1.

satisfactory performance. When the focal length is small, or moderate, this mechanical problem can be solved without great difficulty. For large focal lengths the difficulties, however, would be very great. In fact, astrograph objectives of these types have not up till now been constructed with a focal length of more than about 4 m.

In recent years highly successful constructions of astrograph lens objectives have been added to the Petzval and Cooke types, of which especially the Ross(2) and Sonnefeld(3) quadruplet lens objectives should be mentioned. For the same aperture ratio and field these objectives give a better image quality than the older types. With the latter they share the drawback discussed above of relatively great distance between front and rear lens, and the necessity of very delicate and stable adjustment of the component lenses. Consequently constructions of objectives with a large focal length would meet with the same difficulties as before.

The construction by B. Schmidt(4) of an aplanatic anastigmatic mirror camera constituted an extremely valuable addition to the existing astrograph constructions. When astrographs of long focal length are considered, however, it is a marked drawback of the Schmidt camera that its total length is twice the focal length.

In view of this situation it was considered useful to examine the possibilities of a lens objective which did not suffer from the drawbacks of excessive mutual distances of the optical components and the corresponding requirement of delicate adjustment of mutual position, nor from the drawback of excessive total length of the camera as a whole. It seemed possible to reach a solution of the problem by considering a lens system,

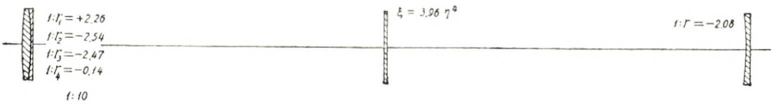

Fig. 2.

the components of which are an achromatic lens pair, a correcting plate of the same type as that used in the Schmidt camera, and a field-flattening lens situated immediately in front of the photographic plate. A lens system, in other words, which results when an achromatic lens pair is corrected with the aid of optical components of the same type as those which convert the spherical mirror into the aplanatic anastigmatic Schmidt camera system.

The investigations lead to the lens system illustrated in Fig. 2, consisting of an achromatic lens pair with specified radii, a correcting plate of specified curve situated in the middle between the lens pair and the photographic plate, and a field-flattening lens, also specified, immediately in front of the photographic plate.

It should be emphasized that, although there is a very considerable distance between the various components of the lens system, this is without consequence, since small displacements of the correcting plate and the field-flattening lens have a negligible effect on the quality of the images.

It follows that with the lens system just described the construction of astrographs of a long focal length presents no greater difficulty than the construction of an astrograph with a Fraunhofer objective of the same focal length and aperture.

Since the price of a correcting plate and a field-flattening lens is considerably lower than that of a Fraunhofer objective of the same diameter, the price of the whole lens system should not be much higher than that of a Fraunhofer objective, again of the same aperture and focal length.

The analysis leading to the lens system just described is given in the following paragraphs. The investigations were carried out in 1940—41. Brief summaries were published shortly afterwards, [(5)] and [(6)]. In 1941 detailed specifications of an astrograph provided with the lens system shown in Fig. 2, p. 6, were sent to Carl Zeiss, Jena, and an order for such an instrument placed. However, it never became possible for the firm mentioned to construct and deliver the instrument.

2. Let us consider the geometrical third order aberrations of an achromatic lens pair consisting of one crown glass and one flint glass lens, with spherical surfaces, assumed to be infinitely thin and in contact. Rotational symmetry about an optical axis is assumed.

The following notation is adopted, cf. [(7)]

n_1 Index of refraction of the crown glass lens

n_2 Index of refraction of the flint glass lens

r_1, r_2 Radius of curvature of the front surface of the crown glass, resp. flint glass lens, positive when convex toward the incident rays

r'_1, r'_2 Radius of curvature of the rear surface of the crown glass, resp. flint glass lens, same sign convention

v Angle of meridional ray incident on the first refracting surface with optical axis, positive when a counterclockwise turn will bring the direction of the incident ray to coincide with that of the optical axis

h Distance of intersection point of incident meridional ray and first refracting surface from the optical axis, positive above the axis

y Distance of intersection point of meridional ray refracted by optical system and focal plane from optical axis, positive above the axis

f Focal length of optical system

A Coefficient of third order spherical aberration
B Coefficient of third order coma
C Coefficient of third order astigmatism
D Coefficient of third order curvature of field
E Coefficient of third order distortion

Consider a meridional ray characterized by h and v (cf. above). The Gaussian image defined by corresponding paraxial rays ($h \to 0$) with angle of incidence v is located in the Gaussian focal plane. Its distance from the optical axis, counted positive above the axis, is denoted by y_0 and is equal to $-f \tan v$. According to the theory of third order, or Seidel aberrations [cf. f. inst. [1] and [8]], the total aberration, i. e. the deviation in the focal plane

$$\Delta = y - y_0 \tag{1}$$

from the Gaussian image, is given by the following expression

$$\Delta = Ah^3 - 3Bh^2v - \frac{1}{2}(C-D)\,hv^2 - Ev^3. \tag{2}$$

We use here the aberrational coefficients A, B, C, D, and E as defined by A. Danjon and A. Couder[9], cf. also [7]. The relations between these coefficients and the coefficients of K. Schwarzschild[1], B_S, F_S, C_S, D_S, and E_S, are as follows,

$$\begin{aligned} A &= -B_S \\ B &= -F_S \\ C &= +2\,C_S \\ D &= -2\,(C_S + D_S) \\ E &= -E_S. \end{aligned}$$

The radii of curvature ϱ_s and ϱ_t of the sagittal and tangential focal plane are, with these definitions,

$$\frac{1}{\varrho_s} = D + C$$

$$\frac{1}{\varrho_t} = D - C.$$

The Petzval radius of curvature ϱ_P is given by

$$\frac{1}{\varrho_P} = 2\,C + D = -\sum \frac{\varphi}{n}.$$

The radius of curvature of a focal surface is counted positive when the surface is convex towards the incident rays, i. e. the same sign convention is used as for the lens surfaces.

For a lens system consisting of two infinitely thin lenses in contact the Seidel coefficients are given by simple relations [cf. f. inst. (1)]. We introduce

$$\left.\begin{aligned} \varphi_1 &= (n_1 - 1)\left(\frac{1}{r_1} - \frac{1}{r_1'}\right) \\ \varphi_2 &= (n_2 - 1)\left(\frac{1}{r_2} - \frac{1}{r_2'}\right) \end{aligned}\right\} \quad (3)$$

and

$$\left.\begin{aligned} \sigma_1 &= (n_1 - 1)\left(\frac{1}{r_1} + \frac{1}{r_1'}\right) \\ \sigma_2 &= (n_2 - 1)\left(\frac{1}{r_2} + \frac{1}{r_2'}\right), \end{aligned}\right\} \quad (4)$$

further

$$\left.\begin{aligned} \gamma_1 &= \frac{\sigma_1}{\varphi_1} = \frac{\dfrac{1}{r_1} + \dfrac{1}{r_1'}}{\dfrac{1}{r_1} - \dfrac{1}{r_1'}} \\ \gamma_2 &= \frac{\sigma_2}{\varphi_2} = \frac{\dfrac{1}{r_2} + \dfrac{1}{r_2'}}{\dfrac{1}{r_2} - \dfrac{1}{r_2'}}. \end{aligned}\right\} \quad (5)$$

Then with

$$\frac{P_1}{\varphi_1^3} = \frac{n_1^2}{8\,(n_1 - 1)^2} + \frac{3\,n_1 + 2}{8\,n_1} - \frac{n_1 + 1}{2\,n_1\,(n_1 - 1)}\,\gamma_1 + \frac{n_1 + 2}{8\,n_1\,(n_1 - 1)^2}\,\gamma_1^2 \quad (6)$$

and

$$\frac{P_2}{\varphi_2^3} = \frac{n_2^2}{8\,(n_2 - 1)^2} + \frac{3\,n_2 + 2}{2\,n_2}\left(\frac{\varphi_1}{\varphi_2} + \frac{1}{2}\right)^2 - \frac{n_2 + 1}{n_2\,(n_2 - 1)}\left(\frac{\varphi_1}{\varphi_2} + \frac{1}{2}\right)\gamma_2 +$$

$$+ \frac{n_2 + 2}{8\,n_2\,(n_2 - 1)^2}\,\gamma_2^2 \quad (7)$$

the coefficient A of spherical aberration is found from

$$-A = P_1 + P_2. \tag{8}$$

Similarly with

$$\frac{Q_1}{\varphi_1^2} = -\frac{2n_1+1}{4n_1} + \frac{n_1+1}{4n_1(n_1-1)}\gamma_1 \tag{9}$$

and

$$\frac{Q_2}{\varphi_2^2} = -\frac{2n_2+1}{2n_2}\left(\frac{\varphi_1}{\varphi_2}+\frac{1}{2}\right) + \frac{n_2+1}{4n_2(n_2-1)}\gamma_2 \tag{10}$$

the coefficient B of coma is obtained from

$$-B = Q_1 + Q_2. \tag{11}$$

Further, for the lens system considered, the coefficient C of astigmatism is

$$C = 1, \tag{12}$$

the coefficient D of curvature of field is

$$-D = 2 + \frac{\varphi_1}{n_1} + \frac{\varphi_2}{n_2}, \tag{13}$$

and the coefficient E of distortion is

$$E = 0. \tag{14}$$

Now, for an achromatic lens pair of the kind considered the refractive powers φ_1 and φ_2 of the crown glass and flint glass, respectively, are determined by the focal length of the instrument and the glass dispersions, specified in the customary way by the optical constants ν_1 and ν_2. In fact

$$\left.\begin{aligned} \varphi_1 &= \frac{\nu_1}{\nu_1-\nu_2}\frac{1}{f} \\ \varphi_2 &= -\frac{\nu_2}{\nu_1-\nu_2}\frac{1}{f}. \end{aligned}\right\} \tag{15}$$

When the optical constants of the glass, viz. n_1, n_2, ν_1, and ν_2 are given, the Seidel coefficients A and B can thus be calculated according to equations (6) to (11) as functions of the lens parameters γ_1 and γ_2 [cf. equation (5)], while the Seidel coefficients C, D, and E are calculable constants.

Suppose now that an infinitely thin correcting plate, of refracting index n_3, is inserted between the achromatic lens pair and the focal plane (cf. Fig. 2, p. 6). The front surface of the correcting plate is assumed to be plane, while the equation of the rear surface is supposed to be

$$\xi = g\eta^4. \tag{16}$$

The constant g measures the deformation of the correcting plate.

At the distance η from the optical axis the correcting plate acts as a prism of refracting angle $4\,g\eta^3$. The angular deviation of a ray penetrating the correcting plate at distance y from the optical axis is, therefore, within the accuracy of third order optics equal to $4\,(n_3-1)\,g\eta^3$, or

$$G\eta^3, \tag{17}$$

with

$$G = 4\,(n_3-1)\,g. \tag{18}$$

When g and G are positive, the thickness of the correcting plate increases with the distance from the optical axis. A ray penetrating the correcting plate at a positive distance η from the optical axis, i. e. above the axis will then be deviated by a counterclockwise rotation.

Let us consider a meridional ray specified by v and h (cf. p. 7). This ray penetrates the lens pair at the distance h from the optical axis, and the focal plane at a distance y from the optical axis, which within the accuracy of first order optics is equal to $-vf$ (cf. Fig. 1, p. 5).

We put the unit of length equal to the focal length, so that $f=1$. The distance from the (infinitely thin) correcting plate to the focal plane is denoted by a. Hence the distance from the achromatic lens pair to the correcting plate is $1-a$.

It is then found that within the accuracy of first order optics the distance η from the optical axis at which the meridional ray specified by v and h penetrates the correcting plate, is given by

$$\eta = h-(1-a)\ (v+h) \tag{19}$$

or

$$\eta = ah-(1-a)\,v. \tag{20}$$

It follows from (17) and (20) that the deviation of the ray considered by the correcting plate is equal to

$$G\left[ah-(1-a)\,v\right]^3 \tag{21}$$

or

$$Ga^3h^3-3\,Ga^2(1-a)\,h^2v+3\,Ga\,(1-a)^2hv^2-G\,(1-a)^3v^3. \tag{22}$$

The deviation causes a shift of position of the point at which the meridional ray considered penetrates the focal plane. The corresponding change Δy of the distance y of the point in question from the optical axis is equal to the angular deviation given by equation (22) multiplied by the distance a from the correcting plate to the focal plane. The quantity Δy has the same sign as the quantity given by (22). Hence

$$\Delta y = Ga^4h^3-3\,Ga^3(1-a)\,h^2v+3\,Ga^2(1-a)^2hv^2-Ga\,(1-a)^3v^3. \tag{23}$$

It is easily seen that equation (23) is correct within the accuracy of third order optics.

Comparing now equation (23) with equation (2) giving the total aberration in terms of the Seidel coefficients, we find that the insertion of the correcting plate considered gives rise to the following changes of the Seidel coefficients:

$$\left.\begin{aligned} \Delta A &= +Ga^4 \\ \Delta B &= +Ga^3(1-a) \\ \Delta\left(\frac{1}{2}C-\frac{1}{2}D\right) &= -3\,Ga^2(1-a)^2 \\ \Delta E &= +Ga\,(1-a)^3. \end{aligned}\right\} \tag{24}$$

According to Petzval's theorem [cf. f. inst. (1), (7), (8), or (9)] we have

$$2\,C+D = -\left(\frac{\varphi_1}{n_1}+\frac{\varphi_2}{n_2}\right). \tag{25}$$

The correcting plate gives no contribution to the right-hand side of equation (25), so that

$$\Delta(2\,C+D) = 0\,. \tag{26}$$

Combining this with the third equation (24), we finally get

$$\left.\begin{aligned} \Delta C &= -2\,Ga^2(1-a)^2 \\ \Delta D &= +4\,Ga^2(1-a)^2. \end{aligned}\right\} \tag{27}$$

Returning now to the expressions for the Seidel coefficients of the lens pair alone, we find [cf. equation (12)] that, in order to correct the astigmatism of the lens pair, the position and form of the correcting plate must be chosen in such a way that

$$1-2\,Ga^2(1-a)^2 = 0$$

or

$$G = \frac{1}{2\,a^2(1-a)^2}. \tag{28}$$

It then follows from the two first equations (24) that the coefficients A and B of spherical aberration and coma through the introduction of the correcting plate are changed by

$$\Delta A = +\frac{1}{2}\left(\frac{a}{1-a}\right)^2$$

and

$$\Delta B = +\frac{1}{2}\left(\frac{a}{1-a}\right). \tag{29}$$

We can now write down the equations expressing the condition that the introduction of the correcting plate, in addition to annihilating the astigmatism, should also annihilate the spherical aberration and the coma of the lens pair, as given by equations (8) and (11), namely,

$$-(P_1+P_2)+\frac{1}{2}\left(\frac{a}{1-a}\right)^2 = 0 \tag{30}$$

and

$$-(Q_1+Q_2)+\frac{1}{2}\left(\frac{a}{1-a}\right) = 0, \tag{31}$$

where P_1, P_2, Q_1, and Q_2 are given by equations (6), (7), (9), and (10).

Considering the quantities n_1, n_2, ν_1, and ν_2 specifying the optical glass of lens pairs as fixed constants of the problem, and remembering that φ_1 and φ_2 are given by equation (15), we see that (30) and (31) can be regarded as two equations which determine γ_1 and γ_2 [cf. equation (5)] as functions of the parameter a measuring the relative distance of the correcting plate from the focal plane.

When γ_1 and γ_2 are known, σ_1 and σ_2 can be found, since φ_1 and φ_2 are known from equation (15). With the aid of equations (3) and (4) we can then find the radii of the optical surfaces of the lens pair, *viz.* r_1, r_1', r_2, and r_2'.

Equation (28) determines G as a function of the parameter a. Equation (18) then fixes g, n_3 being a specified optical constant. Thus the form of the surfaces of the correcting plate is also determined.

The system consisting of the achromatic lens pair and the correcting plate is free from spherical aberration, coma, and astigmatism. The only remaining third order aberration—apart from distortion, which does not give rise to any blurring of the image—is curvature of field. This can be removed without changing any of the other aberrations by the well-known method of placing an infinitely thin field-flattening lens immediately in front of the focal plane.

The coefficient of curvature of field is [cf. equations (13), (27), and (28)]

$$D = -2 - \frac{\varphi_1}{n_1} - \frac{\varphi_2}{n_2} + 2 = -\left(\frac{\varphi_1}{n_1} + \frac{\varphi_2}{n_2}\right). \tag{32}$$

According to a well-known formula [cf. f. inst. [9]] the refracting power φ_4 of the field-flattening lens (of refracting index n_4) necessary to obtain a flat field. is

$$\varphi_4 = n_4 \cdot D \tag{33}$$

or, according to (32),

$$\varphi_4 = -n_4\left(\frac{\varphi_1}{n_1} + \frac{\varphi_2}{n_2}\right). \tag{34}$$

This result can be obtained immediately from the Petzval condition for the whole system consisting of the achromatic lens pair, the correcting plate, and the field-flattening lens, the refracting power φ_3 of the correction plate being equal to zero, *viz.*

$$\frac{\varphi_1}{n_1} + \frac{\varphi_2}{n_2} + \frac{\varphi_4}{n_4} = 0. \tag{35}$$

The coefficient of distortion of the optical system consisting of the lens pair and the correcting plate is given by equation (14), the last equation (24), and equation (28),

$$E = \frac{1-a}{a}. \tag{36}$$

The field-flattening lens introduces an additional distortion

$$E = \frac{\varphi_4}{2\,n_4} + \frac{\varphi_4^2}{2\,n_4\,(n_4-1)}. \tag{37}$$

From equation (36) we infer that the distortion introduced by the correcting plate is large if the plate is placed at a small distance from the focal plane.

3. The next step in the analysis consists in an investigation of the equations (30) and (31), to see whether they have real solutions leading to optical systems of practical value, i.e. for which the curvatures of the lens surfaces and the deformation of the correcting plate are not excessive.

Let us consider, first, the case that the crown glass and flint glass of the achromatic lens pair is the schematic typical glass adopted by K. Schwarzschild, *viz.* crown glass with $n_1 = 1.5$, $\nu_1 = 60$, and flint glass with $n_2 = 1.6$, $\nu_2 = 36$. Equations (15) give for this case, with $f = 1$, $\varphi_1 = +2.5$, and $\varphi_2 = -1.5$. Inserting then numerical values in equations (6), (7), (9), and (10), we find

$$P_1 + P_2 = +13.28 - 26.05\,\gamma_1 + 18.23\,\gamma_1^2 - 10.665\,\gamma_2 - 2.636\,\gamma_2^2 \tag{38}$$

and

$$Q_1 + Q_2 = -0.72 + 5.21\,\gamma_1 + 1.52\,\gamma_2. \tag{39}$$

Equations (30), and (31),

$$P_1 + P_2 - \frac{1}{2}\left(\frac{a}{1-a}\right)^2 = 0 \tag{40}$$

$$Q_1 + Q_2 - \frac{1}{2}\left(\frac{a}{1-a}\right) = 0, \tag{41}$$

can now be solved numerically for any specified value of the parameter a.

The following table gives γ_1 and γ_2 as functions of a. The table further contains the numerical values of the lens curvatures r_1, r_1', r_2, and r_2' together with the numerical value of G, calculated according to the equations enumerated on $p.$ 14

a	0.2	0.4	0.5	0.6	0.8	Ordinary Fraunhofer lens
γ_1	−0.276	−0.174	−0.096	+0.023	+0.642	
γ_2	+1.502	+1.289	+1.132	+0.888	−0.411	
r_1	+1.81	+2.06	+2.26	+2.56	+4.10	+1.67
r_1'	−3.19	−2.94	−2.74	−2.44	−0.90	−3.33
r_2	−3.13	−2.86	−2.66	−2.36	−1.76	−3.26
r_2'	−0.63	−0.36	−0.16	+0.14	+0.74	−0.76
G	19.53	8.68	8.00	8.68	19.53	
g	9.77	4.34	4.00	4.34	9.77	

The table shows that the curvatures are not excessive, in fact for $a = 0.5$ they are about equal to those of an ordinary Fraunhofer lens, or a Petzval lens, and considerably smaller than those of a Cooke triplet lens. It may be noted that the curvatures of the rear surface of the front lens and the front surface of the rear lens are about equal, as is the case of the ordinary Fraunhofer lens.

In order to judge the amount of deformation of the correcting plate corresponding to a given value of G it is convenient to make a comparison with the correcting plate of a Schmidt camera of equal focal length. The deformation constant g_S of a Schmidt plate is given by [cf. [8]]

$$g_S = \frac{1}{32}\frac{1}{n-1}. \tag{42}$$

Now, according to equation (18)

$$g = \frac{1}{4}\frac{1}{n-1}G.$$

Thus the deformation constant of the correcting plate of the present lens system is $8\,G$ times that of a correction plate of a Schmidt camera of equal focal length. Now, according to equation (28) the ratio $8\,G$ is equal to

$$\frac{4}{a^2(1-a)^2}. \tag{43}$$

Further according to equation (16) the deformation constant varies inversely to the third power of the focal length. This means that the deformation of the correcting plate of the present lens system is equal to that of a Schmidt camera of the same aperture, but with a focal length that is smaller in the ratio of

$$(8\,G)^{\frac{1}{3}} = \left(\frac{4}{a^2\,(1-a)^2}\right)^{\frac{1}{3}} \tag{44}$$

to 1. The correcting plate hence corresponds to that of a Schmidt camera, which has an aperture ratio $(8\,G)^{\frac{1}{3}}$ times greater than that of the lens system considered.

It is seen that G has a minimum for $a = 0.5$, with $G = 8$, and $(8\,G)^{\frac{1}{3}} = 4$. It is thus perfectly practicable to make the correcting plate of a lens system of the type considered for the aperture ratio 1:10, or even 1:5.

It follows from the discussion on p. 21 that it is not desired to construct the lens system in question with a very great aperture ratio. This may be noted in connection with the above discussion regarding the degree of deformation of the correcting plate.

In the following sections we shall give the results of calculations leading to the specification of lens systems with lenses and correcting plate of finite thickness. We shall also state the results of calculations—according to the method of trigonometrical tracing of rays—showing the state of correction of the lens systems in question. All these calculations have been restricted to the case of a correcting plate situated half-way between the achromatic lens pair and the focal plane, i. e. $a = 0.5$.

4. According to the procedure described above it is possible to determine the geometrical parameters of a lens system consisting of an infinitely thin achromatic lens pair, an infinitely thin correcting plate, and an infinitely thin field-flattening lens in such a way that the whole system is achromatic, aplanatic (i. e. $A = 0$, $B = 0$), anastigmatic ($C = 0$), and with a flat field ($D = 0$), in other words so that all third order aberrations (except distortion) vanish.

In order to prepare the actual construction of such an instrument it is necessary to modify the geometrical parameters in such a way that the lenses and the correcting plate have suitable finite thicknesses, while the lens system still satisfies the two conditions of achromatism together with the conditions $A = 0$, $B = 0$, $C = 0$, and $D = 0$.

Starting with the data obtained for infinitely thin lenses and correcting plate it is possible by a process of systematic small variations of the geometrical parameters to arrive at a lens system satisfying the conditions specified. The process includes a number of numerical determinations of the aberration constants A, B, C, and D by using the method of trigonometrical ray tracing [cf. f. inst. (7)]. The final determination of the state of correction of the resulting lens system is also made by the method of trigonometrical ray tracing.

Such calculations have been carried out by Mr. Erik Lorensen, M. Sc. Altogether four different systems were investigated, two of which were achromatized for the photographic wavelength region, while two were corrected for the photovisual. Below we give a summary of the calculations of one of the photovisual lens systems. It should be noted that this system was designed for use in connection with a curved photographic plate, concave towards the incident rays, i. e. without a field-flattening lens.

Optical constants.
Crown glass, first lens, and correcting plate.

$$n_C = 1.50725,\ n_d = 1.50977,\ n_F = 1.51549,\ n_g = 1.51995,\ n_h = 1.52364$$

Flint glass, second lens.

$$n_C = 1.61504,\ n_d = 1.62004,\ n_F = 1.63210,\ n_g = 1.64206,\ n_h = 1.65068$$

Geometrical data for infinitely thin lenses and plate.

$$\varphi_1 = +2.424,\ \varphi_2 = -1.424,\ f = 1,\ \text{for } d\text{-light}.$$

$$\frac{1}{r_1} = +2.244,\ \frac{1}{r_1'} = -2.512,\ \frac{1}{r_2} = -2.443,\ \frac{1}{r_2'} = -0.146,\ g = 3.92.$$

Geometrical data for final lens system, unit $f = 1$.

Axial thicknesses, crown glass lens 0.011, flint glass lens 0.004, correcting plate 0.005. Axial separation of lenses, 0.0005. Axial separation of rear lens, rear surface, and correcting plate, front surface, 0.4853.

$$\frac{1}{r_1} = +2.258\ 8787,\ \frac{1}{r_1'} = -\ 2.536\ 4387,\ \frac{1}{r_2} = -2.473\ 0643,$$
$$\frac{1}{r_2'} = -0.138\ 7457.\ g = 3.960.$$

Residual aberrations, *d*-light.

Third order aberrations for *d*-light. Spherical aberration $0''.00$ O^3, coma $0''.00$ O^2G, astigmatism $0''.00$ OG^2 (cf. p. 3). Curvature of field $\frac{1}{\varrho_P} = -0.726$.

Aberrations of the fifth and higher orders. Spherical aberration for marginal rays with aperture ratio 1:10, $0''.2$. Maximum extent of aberrational image (on Petzval sphere), for *d*-rays with aperture ratio 1:10, and angular diameter of field $5°.7$, $2''.0$.

Secondary spectrum.

The secondary spectrum is practically equal to the secondary spectrum of an ordinary Fraunhofer lens with the same aperture and focal length. When the photographic plate coincides with the focal sphere of *d*-light (λ = 5876 Å), the aberrational disc due to the secondary spectrum has a diameter of $10''O$ (unit of aperture ratio O 1:10, cf. p. 3) for *C*-light (λ = 6563 Å), or *F*-light (λ = 4861 Å). Light within the more restricted wave-length range 5400 Å—6300 Å gives rise to an aberrational disc of diameter $3''O$. The variation of focal length with wave-length is practically the same as that of the distance between the lens pair and the focus, thus also with the ordinary Fraunhofer lens. This means that with an angular diameter of the field equal to $5°.7$, the chromatic elongation of the image in the direction of the center of the field is very small, the maximum extent being $0''.2$ for the wave-length range 5400—6300 Å.

Chromatic differences of third and higher order aberrations.

The third order aberrations for *C*-light, and *F*-light, differ very little from those of *d*-light. The same applies to the aberrations of fifth and higher order. The effects of these chromatic differences are quite inappreciable in comparison with those of the secondary spectrum.

We give, next, a brief summary of the data describing a lens system differing from the one just described only in its being achromatized for the photographic wave-length region. It has been designed in such a way that the correcting plate is identical with that of the photovisual system, but placed somewhat nearer to the focal plane.

The same glass is used as for the previous system. The axial thicknesses and the axial separation of the lenses are also the same. The axial separation of the rear lens, rear suffface, from the correcting plate, front surface, now, however, is 0.5412. The curvatures of the spherical lens surfaces are

$$\frac{1}{r_1} = +2.123, \quad \frac{1}{r'_1} = -2.102, \quad \frac{1}{r_2} = -2.084, \quad \frac{1}{r'_2} = -0.217.$$

The state of correction is very nearly the same as for the previous system.

Finally we give a brief summary of the data referring to a lens system which includes a field-flattening lens. The data refer to the instrument to be made by Carl Zeiss (cf. p. 7) with 15 cm. aperture, and 1.5 m. focal length, photovisual lens system. The state of correction is practically as good. The maximum extent of total aberration, apart from secondary spectrum is again 2″ for the aperture ratio 1:10 and an angular diameter of field 5°.7. The data of the system are

	C-light	d-light	F-light
n_1	1.51508	1.51757	1.52318
n_2	1.61598	1.62103	1.63317
n_3	1.51436	1.51684	1.52244
n_4	1.51534	1.51784	1.52344

Axial thicknesses and separations in mm. First lens, 22.0, first to second lens 0.2, second lens 15.0, second lens to correcting plate 714.0, correcting plate 15.0, correcting plate to field-flattening lens 680.0, field-flattening lens 20.0.

Radii of spherical surfaces in mm., $r_1 = +659.4$, $r'_1 = -655.1$. $r_2 = -665.3$, $r'_2 = -15150.0$. Equation of deformed surface of correcting plate, $\xi = 1.255 \cdot 10^{-9} \eta^4$, with ξ and η in mm. Radius of front surface of field-flattening lens, -719.7 mm., rear surface plane.

5. It appears from the previous discussion that the lens system considered is suitable for astrographs. If we choose the aperture ratio 1:10, the geometrical aberrations are almost negligible within a field of 5°.7 angular diameter, or within a square field 4°.0 × 4°.0. When the correcting plate is given the same diameter as the lens pair, there is no silhouetting. The positions of the correcting plate and the field-flattening lens relative to the lens pair are not critical so that the large separation of the former from the latter is no drawback. It is not necessary that the correcting plate and the field-flattening lens should be perfectly homogeneous with regard to refracting index. In this connection it may be mentioned that, with the aperture ratio and field considered, the maximum angular deviation of a ray produced by the correcting plate is about 200″, giving a deflection in the focal plane equivalent to about 100″.

It should be emphasized that the possibilities of the construction have not been exhausted. It is quite possible that the aberrations of the fifth and higher orders could be reduced by a change of position of the correcting plate and by the introduction of a small curvature of the correcting plate ($\xi = \alpha\eta^2 + g\eta^4$, α small). Thus it might be possible to construct systems of larger aperture ratio and field and yet with the same image quality.

As mentioned on p. 6 the principal application of the lens system is for astrographs with a long focal length, and for such instruments it is not desirable to go much beyond an aperture ratio of 1:10, and a field of 4° × 4°. Let us consider a focal length of, say, 8 m. For photographic work it will not be desirable to go beyond an aperture of about 80 cm., since the light loss through the absorption, especially in the achromatic lens pair, will nearly compensate the light gain through the greater aperture, if the latter is further increased. Also, a field of 4° × 4° has the linear extent of 56 cm. × 56 cm., so that it becomes impracticable to cover the whole field with a single plate. Four plates, about 30 cm. × 30 cm. might be used here to cover the field, but it will not in general be desirable to use a still larger field. Considerations of the conditions necessary for sufficiently exact guiding during exposure lead to a similar conclusion.

6. It is a well-known fact that photographic photometry of faint stars in galactic regions of high star density requires the use of astrographs of a long focal length. With a small focal length the photographic stellar images are not sufficiently separated. This is illustrated by the following considerations.

The magnitude of a star obtained from measurements on a photographic plate is influenced when some other star is situated very close to it. If there is a faint neighbouring star, the magnitude may be influenced though the disturbing star be invisible on the plate. When the disturbing star is very close to the star to be measured, it will be the magnitude corresponding to the combined light that is found from the measurement.

The amount of the disturbance will be a rather complicated function of the magnitude difference and the linear distance between the stars on the photographic plate, and it will depend upon the micro-photometric method used in determining the strength of the photographic image.

Qualitatively it may be said, however, that if the distance between the two stars is less than about 0.05—0.06 mm. there will be a disturbance not much less than that which would result if the stars practically coincided. There will thus be associated with each star to be measured an area of about 0.01 mm.^{2} which must be free from disturbing stars if the photometric magnitude obtained is to be correct. Strictly speaking, the area in question is dependent on the focal length *f*, in such a way that it increases with *f*. Within a certain range of *f* the variation of this area with *f* is not very pronounced, however, since very small images are not suitable for photometric work. In the following discussion we shall neglect the variation in question.

Let the magnitude of the star to be measured be *m*. If an accuracy of $0^m.1$ is aimed at, all stars brighter than about $m + 2^m.5$ must be considered as disturbing stars. If an accuracy of $0^m.02$ is wanted, the corresponding limit is about $m + 4^m.5$.

The number of stars per square degree is a known function of the magnitude and the galactic latitude. The following table has been derived from the tables given by P. J. van RHIJN(10). It gives the number of stars per square degree, per interval of magnitude 1^m at photographic magnitude m, and galactic latitude b. The values given for 19^m, 20^m, and 21^m are extrapolated.

	$b = 0°$	10°	20°	40°	60°	90°
12^m	90	50	30	17	11	10
13	200	120	80	40	20	20
14	600	300	180	80	40	40
15	1500	800	400	150	80	60
16	3000	1900	900	300	150	100
17	8000	4000	1700	500	300	170
18	17000	8000	3000	800	400	300
19	30000	14000	5000	1200	600	400
20	60000	20000	8000	1700	800	600
21	100000	40000	11000	2000	1000	700

The average number of stars per 0.01 mm.^{2} is given by the expression

$$\frac{A}{30500} \frac{1}{f^2}$$

if A is the number of stars per square degree and f the focal length.

The following table gives the average number of stars per 0.01 mm.^{2}, per interval of magnitude 1^m, for focal length 1 m. For other focal lengths the corresponding numbers are obtained by division with the square of the focal length expressed in meters.

	$b = 0°$	10°	20°	40°	60°	90°
12^m	0.003	0.0015	0.0010	0.0005	0.0004	0.0003
13	0.008	0.004	0.003	0.0012	0.0008	0.0007
14	0.02	0.010	0.006	0.002	0.0014	0.0011
15	0.05	0.02	0.013	0.005	0.003	0.002
16	0.11	0.06	0.03	0.009	0.005	0.003
17	0.2	0.13	0.06	0.016	0.008	0.006
18	0.5	0.3	0.10	0.03	0.013	0.010
19	1.0	0.5	0.17	0.04	0.019	0.014
20	1.8	0.8	0.3	0.05	0.03	0.018
21	3	1.2	0.4	0.07	0.03	0.02

Let us consider the case that it is desired to determine photographic magnitudes for stars in a milky-way field, $b = 0°$, of a magnitude around 16^{m}. With focal length 1 m. the average number of disturbing stars between magnitude $17^{m}.5$ and $18^{m}.5$, would be about 0.5, each star giving rise to a photometric error $0^{m}.1 - 0^{m}.2$, and even much greater errors would not be infrequent. To solve the photometric problem in a satisfactory way it will therefore be necessary to use a focal length considerably greater. With $f = 4$ m. the frequency of the errors would, according to the table, be as follows.

$> 0^{m}.24$	2 per cent.
$0\ .10 - 0^{m}.24$	3
$0\ .04 - 0\ .10$	7
$0\ .02 - 0\ .04$	11

$m = 16^{m}$, $b = 0°$, $f = 4$ m.

It depends, of course, upon the particular application in view, whether this is considered satisfactory, or not.

If the object is to determine photographic magnitudes for stars of photographic magnitude 18^{m} in the milky way ($b = 0°$), it will usually be necessary to increase the focal length still further. Even with $f = 8$ m. the photometric errors of the kind considered are not negligible, as is apparent from the following table of the frequency of errors analogous to the one given above.

$> 0^{m}.24$	3 per cent.
$0\ .10 - 0^{m}.24$	3
$0\ .04 - 0\ .10$	5
$0\ .02 - 0\ .04$	8

$m = 18^{m}$, $b = 0°$, $f = 8$ m.

It should be emphasized that the above estimates are necessarily rather rough. They show, however, that for photometric work on faint stars in the milky way it may be desirable to use astrographs of about 8 m. focal length, or more.

If colour indices are determined by comparison of photographic and photovisual magnitudes obtained from separate plates taken with the same instrument, or instruments of the same focal length, then the error of the colour index due to a disturbing star is proportional to the colour index difference between the two stars, and the resulting colour index is intermediate between that of the two stars. The chance that stars of apparently exceptional colour index are fortuitously produced is thus extremely small.

If, however, the photographic and photovisual magnitudes are obtained from two exposures on the same plate, then the errors due to disturbing stars are uncorrelated for the photographic and photovisual magnitudes. The errors of the colour index, therefore, are not limited as in the case considered above, and fortuitous exceptional colour indices will not be quite rare.

These considerations show that particular caution is indicated when the method of obtaining photographic and photovisual magnitudes from one plate is used, or else when more than one exposure is made on each plate. It will usually be necessary, in such cases, to increase the focal length beyond a value which would otherwise have been satisfactory.

When photometry of faint stars in high galactic latitudes is considered, the situation is quite different on account of the much smaller star density.

The following tables of the frequencies of photometric errors which are valid for $b = 90°$, are analogous to those given on p. 24

$< 0^m.24$	0.07 per cent
$0\ .10 - 0^m.24$	0.06
$0\ .04 - 0\ .10$	0.09
$0\ .02 - 0\ .04$	0.11

$m = 16^m, \quad b = 90°, \quad f = 4$ m.

$0^m.24$	0.06 per cent.
$0\ .10 - 0^m.24$	0.03
$0\ .04 - 0\ .10$	0.03
$0\ .02 - 0\ .04$	0.04

$m = 18^m, \quad b = 90°, \quad f = 8$ m.

Here the errors are quite negligible. In fact, since the star density is between 50 and 200 times smaller than in the milky way, focal lengths of about 0.5 m., respectively 1 m., would have given results comparable to, or slightly better than those obtained in the milky way with 4 m., respectively 8 m., focal length.

Now, the limiting magnitude, which is reached when the exposure time is extended to the limit set by the sky-background [cf. [11]], is a function of the focal length. In order to reach the limiting magnitudes 16^m, or 18^m, the focal length must be chosen somewhat greater than 0.5 m., respectively 1 m. This means that, in regions of high galactic latitude, conditions for reducing the photometric errors due to disturbing stars are automatically fulfilled.

If the focal length is chosen equal to 2— 3 m., methods involving double or multiple exposure on one plate may safely be used here. With one-exposure methods investigations can safely be extended to intermediate galactic latitudes.

7. Astrographs intended for photometric and spectroscopic Durchmusterung work may be classified as follows. The most powerful instruments are the large reflectors, but it may sometimes be felt that their field is too small. The Ross correctors [cf. (12)] are a great improvement in this respect, but even their extended field may sometimes be considered too small. The Schmidt cameras fulfil every requirement with respect to aperture and field, but because of the fact that the total length of the construction is twice the focal length, the focal length is somewhat restricted. When faint stars are concerned, this type of astrograph is therefore better suited for work in high and intermediate galactic latitudes than in low ones. Astrographs with triplet, or quadruplet, lenses have applications in much the same field as the Schmidt camera. According to the particular problem in view, a choice will be made between mirror and lens objective. The lens system investigated in the present paper has optical properties similar to those of the triplet and quadruplet constructions, but contrary to the latter it is suitable also for instruments of a great focal length.

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